Some problems in Hamiltonian geometry of PDEs

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Plan of the talk

- Introduction to the Hamiltonian formalism for PDEs
- Geometry of the Hamiltonian formalism
- Classification of bi-Hamiltonian (integrable) PDEs

Hamiltonian mechanics, notation

The Hamilton equations:

$$\begin{pmatrix} q^i_t \\ p_{i,t} \end{pmatrix} = \omega^{ij} \begin{pmatrix} \dfrac{\partial H}{\partial q^j} \\ \dfrac{\partial H}{\partial p_j} \end{pmatrix}$$

The symplectic form $\omega = (\omega_{ij})$: $\omega_{ji} = -\omega_{ij}$ and $d\omega = 0$.

The Poisson tensor $P = (\omega^{ij}) = (\omega_{ij})^{-1}$ has vanishing Schouten bracket: $[P, P] = 0$

The Poisson bracket: $\{f_i, f_j\} = P(df_i, df_j) = \omega^{kh} \dfrac{df_i}{dz^k} \dfrac{df_j}{dz^h}$,
$z^h = q^h$ or $z^h = p_h$.

Liouville integrability: n independent conserved quantities f_i in involution, $\{f_i, f_j\} = 0$.

History of Integrability for PDEs

Integrability for Partial Differential Equations is defined as the existence, for a given equation, of an infinite sequence of symmetries or conserved quantities in involution.

- CS Gardner, JM Greene, MD Kruskal, RM Miura (1967-11-06) *Method for Solving the Korteweg-deVries Equation*. Physical Review Letters. **19** (1967), 1095–1097.
- P Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Applied Math., **21** (5) (1968), 467–490.
- VE Zakharov, LD Faddeev, *Korteweg–de Vries equation: A completely integrable Hamiltonian system*, Funktsional. Anal. i Prilozhen., 5:4 (1971), 18–27; Funct. Anal. Appl., 5:4 (1971), 280–287.

Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\,dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta\mathcal{H}}{\delta u^j}\right), \quad \text{with} \quad \frac{\delta\mathcal{H}}{\delta u^j} = (-1)^\sigma\partial_\sigma\frac{\partial h}{\partial u^j_\sigma}$$

where $A = (A^{ij})$ is a Hamiltonian operator (Poisson tensor), i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), with further properties.

Hamiltonian operators

A is a Hamiltonian operator if and only if

$$\{F, G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

$\{,\}_A$ is a Poisson bracket if and only if:

- A is skew-adjoint: $A^* = -A$, where

$$A^*(\psi)^j = (-1)^\sigma \partial_\sigma \left(A^{ij\sigma} \psi_i \right)$$

- The variational Schouten bracket vanishes:

$$[A, A](\psi^1, \psi^2, \psi^3) =$$
$$2 \left[\frac{\partial A^{ij\sigma}}{\partial u^l_\tau} \partial_\sigma(\psi^1_j) \partial_\tau (A^{lk\mu} \partial_\mu(\psi^2_k)) \psi^3_i + \text{cyclic}(1, 2, 3) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

Example: the Korteweg–de Vries equation

The equation:

$$u_t = uu_x + u_{xxx}$$

The bi-Hamiltonian formalism:

$$A_1 = \partial_x, \qquad A_2 = \frac{1}{3}u_x + \frac{2}{3}u\partial_x + \partial_{xxx}$$

with Hamiltonians:

$$H_1 = \frac{u^3}{6} + \frac{u_x^2}{2}, \qquad H_2 = \frac{u^2}{2}$$

Fundamental discoveries:

- KdV as a Hamiltonian system through A_1 (Zakharov, Faddeev '70);
- KdV as a bi-Hamiltonian system through A_1, A_2 (Magri '78);

Motivation for Hamiltonian PDEs

- A Hamiltonian operator maps *conservation laws* to *symmetries*.
- Two compatible Hamiltonian operators A_1, A_2 generate a sequence of conserved quantities (Magri, JMP 1978):

$$A_1\left(\frac{\delta H_{n+1}}{\delta u^i}\right) = A_2\left(\frac{\delta H_n}{\delta u^i}\right).$$

- Integrability: the above sequence H_1,H_2, …,H_n,… is in involution:

$$\{H_i, H_j\} = 0.$$

- There is **no** analogue of Liouville theorem for PDEs, but integrable nonlinear equations usually are C-integrable or S-integrable (Calogero 1980).
- Bi-Hamiltonian systems and their hierarchy.

What makes bi-Hamiltonian systems integrable?

According with Hitchin, Segal, Ward, in *Integrable Systems: Twistors, Loop Groups, and Riemann Surfaces* (1999), integrable systems are characterized by

- the existence of many conserved quantities;
- the presence of algebraic geometry;
- the ability to give explicit solutions.

Structure of bi-Hamiltonian pairs

We consider a class of Hamiltonian operators that are manifestly differential-geometric invariant, homogeneous Hamiltonian operators.

A wide class of known bi-Hamiltonian systems have their Hamiltonian operators in the form of linear combination of homogeneous Hamiltonian operators with different homogeneity degrees:

$$A = A_1 + \epsilon A_2 + \epsilon^2 A_3 + \dots$$

An extension to an infinite formal sum is a building block of Dubrovin–Zhang's perturbative approach to the classification of Integrable Systems.

First-order homogeneous operators

First-order homogeneous operators were introduced in 1983 by Dubrovin and Novikov:

$$\boxed{P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k}$$

They are form-invariant with respect to point transformations of the type:

$$\bar{u}^i = U^i(u^j).$$

where $u^i = u^i(t,x)$, $i, j = 1,\ldots,n$ (n-components).

Homogeneity: $\deg \partial_x = 1$.

Differential geometry and homogeneity

We work in the non-degenerate case $\det(g^{ij}) \neq 0$. Let $(g_{ij}) = (g^{ij})^{-1}$.

After a point transformation $\bar{u}^i = U^i(u^j)$:

- $g^{ij}(\mathbf{u})$ transforms as a contravariant 2-tensor;
- Then
$$\Gamma^i_{jk} = -g_{jp} b^{pi}_k$$
transform as the Christoffel symbols of a linear connection.

Differential geometry and the Hamiltonian property

Skew-adjointness is equivalent to:

- symmetry of g^{ij};
- the connection Γ is metric: $\nabla[\Gamma]g = 0$;

Jacobi identity holds iff:

- the connection Γ is symmetric: $\Gamma^i_{jk} = \Gamma^i_{kj}$, hence it is the Levi–Civita connection of (g_{ij});
- the pseudo-Riemannian metric (g_{ij}) is flat.

It turns out that the operator P_1 admits the canonical form

$$P_1^{ij} = \eta^{ij}\partial_x, \qquad \eta^{ij}: \text{ a constant matrix.}$$

Higher-order homogeneous operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We consider here second-order and third-order homogeneous operators:

$$
\begin{aligned}
R_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\
&+ c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m,
\end{aligned}
$$

$$
\begin{aligned}
R_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\
&+ [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\
&+ d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n.
\end{aligned}
$$

Differential geometry and homogeneity

We will work in the non-degenerate case $\det(g_k^{ij}) \neq 0$, $k = 1, 2$.

After a point transformation $\bar{u}^i = U^i(u^j)$:

- g_2^{ij}, g_3^{ij} transform as contravariant 2-tensors;
- $\Gamma_{2\ jk}^{i} = -g_{jp}c_{2\ k}^{pi}$ and $\Gamma_{3\ jk}^{i} = -g_{jp}c_{3\ k}^{pi}$ transform as linear connections.

Differential geometry and the Hamiltonian property

It was proved (Potëmin, 1992; Doyle, 1992) that the Hamiltonian property implies that

- $\Gamma^{i}_{2\ jk}$ and $\Gamma^{i}_{3\ jk}$ are symmetric and flat;
- in flat coordinates, we have

$$R_2 = \partial_x(g_2^{ij})\partial_x;$$

$$R_3 = \partial_x(g_3^{ij}\partial_x + c_{3\ k}^{ij} u_x^k)\partial_x.$$

The Hamiltonian property

R_2: $g_{2\ ij} = T_{ijk}u^k + T_{0jk}$, where T is completely skew-symmetric and constant;

R_3: let $c_{ijk} = g_{3iq}g_{3jp}c^{pq}_{3k}$; the following properties hold:

$$g_{3mn,k} = -c_{mnk} - c_{nmk}, \qquad c_{nkm} = \frac{1}{3}(g_{3nm,k} - g_{3nk,m}),$$

$$g_{3mn,k} + g_{3nk,m} + g_{3km,n} = 0,$$

$$c_{mnk,l} = -g_3^{pq}c_{pml}c_{qnk}.$$

Further differential geometry for R_3

After a point transformation $\bar{u}^i = U^i(u^j)$:

- $c^i_{jk} = g_{3js} c^{si}_{3k}$ transforms as a linear connection;
- c^i_{jk} is a purely torsion connection;
- c^i_{jk} is flat: $c^i_{jk,l} = c^i_{pl} c^p_{jk}$;
- c^i_{jk} is metric: $\nabla[c] g_3 = 0$;
- c^i_{jk} is in the first category of Cartan's classification of flat metric connections with torsion T, namely

$$T^{ijk} + T^{kij} + T^{jki} = 0, \qquad i \neq j \neq k \neq i,$$
$$T^{ki}_k = 0.$$

The converse does not hold.

Classification of Hamiltonian or bi-Hamiltonian equations

Classification programs of distinguished Integrable Systems depend on the choice of the group and the group action. Usually, the group of point transformations is too small to yield few equivalence classes.

- Classifying bi-Hamiltonian pairs is a way to classify the corresponding integrable hierarchies.
- Bi-Hamiltonian pairs of first-order homogeneous Hamiltonian operators P_1, P_2 are impossible to classify: there are just too many of them.

- Miura group is used by Dubrovin and Zhang in their perturbative approach:

$$\tilde{u}^i = f^i(u^j) + \epsilon F_1^i(u^j)_k u_x^k + \epsilon^2 (F_2^i(u^j)_k u_{xx}^k + G_2^i(u^j)_{kh} u_x^k u_x^h) + \cdots$$

Regarding bi-Hamiltonian pairs of operators of the form

$$A = A_1 + \epsilon A_2 + \epsilon^2 A_3 + \ldots$$

as deformations of first-order pairs, equivalence classes under Miura group are uniquely determined by their dispersionless limit and by n functions of one variable, the central invariants (Liu, Zhang (2005), Carlet, Posthuma, Shadrin (2015)).

- Reciprocal transformations are nonlocal transformations of the independent variable:

$$d\tilde{x} = \Delta(u^i, u^i_x, u^i_{xx}, \dots)\, dx.$$

  They have been used to show that certain quasilinear first-order systems in gas dynamics can be linearized (Rogers, 1968).
- The two types of transformation can be combined into Miura-reciprocal transformations. Dubrovin–Zhang classification scheme was extended to Miura-reciprocal transformations (Lorenzoni, Shadrin, V. 2023) with similar results.

Bi-Hamiltonian systems from finite combinations of homogeneous Hamiltonian operators

Two main mechanisms:

- Compatible triples (*regular mechanism*):
  - with third-order operators: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..
  - with second-order operators: AKNS, 2-component Camassa-Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- Compatible pairs (*singular mechanism*):
  - with third-order operators: Monge–Ampère, WDVV, Oriented Associativity (or F-manifolds) equation (as quasilinear systems of the first order);
  - with second-order operators: new systems here!

Bi-Hamiltonian systems from compatible triples

Many bi-Hamiltonian systems are indeed compatible triples of Hamiltonian operators P_1, Q_1, R introduced by Olver and Rosenau (1996):

$$A_1 = P_1, \qquad A_2 = Q_1 + \epsilon^i R_i \qquad \text{where}$$

$$[R_i, P_1] = 0, \quad [R_i, Q_1] = 0, \quad [P_1, Q_1] = 0, \quad i = 2, 3.$$

Examples:

- with second-order operators R_2: AKNS, 2-component Camassa-Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- with third-order operators R_3: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..

Examples of compatible triples

A classification of bi-Hamiltonian hierarchies which are defined by a triple of mutually compatible Hamiltonian operators was provided by Lorenzoni, Savoldi, V. (JPA 2017).
Examples: scalar case. We have one third-order operator R_3, two first order operators P_1, Q_1:

$$[R_3, P_1] = [R_3, Q_1] = [P_1, Q_1] = 0$$
$$P_1 = \partial_x, \qquad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3.$$

KdV hierarchy (Magri (1978)):

$$\Pi_\lambda = Q_1 + \epsilon^2 R_3 - \lambda P_1 = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3$$

Camassa–Holm hierarchy:

$$\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_3) = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3).$$

Example: the 2-component case

We have one second-order operator R_2 and two first-order operators P_1, Q_1, all of them mutually compatible:

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}$$

Example: the 2-component case

- $\Pi_\lambda = Q_1 + \epsilon^2 R_2 - \lambda P_1$ AKNS (or two-boson) hierarchy, starting from the system of PDEs

$$u_t = (2uv)_x - u_{xx},$$
$$v_t = 2vv_x - 2u_x + v_{xx};$$

- $\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 R_2)$ two-component Camassa-Holm hierarchy (Liu and Zhang, 2005).

Classification by projective reciprocal transformations

Consider a reciprocal transformations of projective type:

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = S^i(u^j) = (S^i_j u^j + S^i_0)/\Delta$$

where $\Delta = S^0_j u^j + S^0_0$. Then,

- R_2 and R_3 transform into new second-order and third-order homogenous Hamiltonian operators;
- P_1 (or Q_1) transform into new non-local first order homogeneous Hamiltonian operators (Ferapontov 1991):

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + u^i_x \partial_x^{-1} w^j_k u^k_x + w^i_h u^h_x \partial_x^{-1} u^j_x$$

Classification by reciprocal projective transformations

The algorithm:

- Second-order and third-order homogeneous Hamiltonian operators can be classified by means of reciprocal projective transformations in low ($\leqslant 8$) dimensions.
- Fix a leading order operator in canonical form R;
- Find all first-order homogeneous Hamiltonian operators P that are compatible with R:

$$[P, R] = 0.$$

Homogeneous Hamiltonian operators and algebraic geometry

New results (Ferapontov, Pavlov, V., JGP 2014, IMRN 2016; Vergallo, V. Nonlinearity 2023).

- Second order homogeneous Hamiltonian operators R_2 are in bijective correspondence with linear line congruences;
- Third order homogeneous Hamiltonian operators R_2 are in bijective correspondence with quadratic line complexes.

Digression: Plücker's line geometry

Two infinitesimally close points $V, V + dV \in \mathbb{P}(\mathbb{C}^{n+1})$,

$$V = [v^1, \ldots, v^{n+1}], \quad V + dV = [v^1 + dv^1, \ldots, v^{n+1} + dv^{n+1}]$$

define a line with coordinates

$$p^{\lambda\mu} = v^\lambda dv^\mu - v^\mu dv^\lambda = \det \begin{pmatrix} v^\lambda & v^\mu \\ v^\lambda + dv^\lambda & v^\mu + dv^\mu \end{pmatrix}$$

inside the projective space: $\mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$ (S. Lie coordinates for Plücker embedding).
We regard (u^i), $i = 1, \ldots, n$ as an affine chart on $\mathbb{P}(\mathbb{C}^{n+1})$, so that $u^{n+1} = 1$, $du^{n+1} = 0$ and

$$p^{ij} = u^i du^j - u^j du^i, \quad p^{(n+1)i} = du^i.$$

Canonical forms of homogeneous Hamiltonian operators

In the non-degenerate case ($\det(g^{ij}) \neq 0$) the second and third order operators admit canonical forms by means of a point transformation (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95)

$$R_2^{ij} = \partial_x \circ g_2^{ij} \circ \partial_x,$$

$$R_3^{ij} = \partial_x \circ (g_3^{ij}\partial_x + c_{3\,k}^{ij} u_x^k) \circ \partial_x,$$

The canonical forms are invariant with respect to projective reciprocal transformations.

The algebraic variety of R_2

(Vergallo, V., Nonlinearity '23) The second-order operator R_2 yields the three-form

$$\omega_2 = (g_2)_{ij} du^0 \wedge du^i \wedge du^j, \qquad (g_2)_{ij} = (g_2^{ij})^{-1}.$$

Intersecting the linear system $i_L\omega_2 = 0$ with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a linear line congruence, an algebraic variety of dimension $n-1$:

$$X_{\omega_2} = \mathbb{G}(2, \mathbb{C}^{n+1}) \cap \{i_L\omega_2 = 0\} = \{\omega_{\lambda\mu\nu} p^{\lambda\mu} = 0\}.$$

It is remarkable that they are Fano varieties (of index 3).

The algebraic variety of R_3

(Ferapontov, Pavlov, V., JGP 2014, IMRN 2016) The third-order operator R_3 fulfills the condition:

$$\partial_i (g_3)_{jk} + \partial_k (g_3)_{ij} + \partial_j (g_3)_{ki} = 0.$$

It implies that g_3 is a Monge metric: a quadratic form in Plücker's coordinates

$$g_3 = X^T Q X = f_{\lambda\mu,\rho\sigma} p^{\lambda\mu} p^{\rho\sigma}.$$

Intersecting g_3 with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a quadratic line complex.

Projective classification of triples

Initiated in Lorenzoni, Savoldi, V. JPA 2017.
Here we classify triples

$$A_1 = P_1 + R_2, \qquad A_2 = Q_1,$$

where

- R_2 is a second-order operator:

$$R_2 = \eta^{ij}\partial_x^2, \quad \text{where} \quad \eta^{ij} = -\eta^{ji}, \quad \det(\eta^{ij}) \neq 0;$$

- P_1, Q_1 are Ferapontov operators of localizable type:

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + w^i_k u^k_x \partial_x^{-1} u^j_x + u^i_x \partial_x^{-1} w^j_k u^k_x$$

Consequences of compatiblity

The compatibility of the Hamiltonian operators: $[P_1, R_2] = 0$ implies the conditions:

- Γ^{ij}_k defines a Frobenius algebra structure on the tangent space of the field variables;
- η_{ij} and Γ^{ij}_k define a cyclic Frobenius algebra (Buchstaber, Mikhailov 2023).
- Set $\bar{g}_{ab} = \eta_{jb}\eta_{ia}g^{ij}$. One of the conditions is:

$$\bar{g}_{bc,a} + \bar{g}_{ca,b} + \bar{g}_{ab,c} = 0,$$

hence $\bar{g}_{ab}$ is the Monge form of a quadratic line complex: it can be rewritten as a quadratic form of $p^{\lambda\mu} = u^{\lambda}du^{\mu} - u^{\mu}du^{\lambda}$.

Example: Kaup–Broer system

Kupershmidt '85. The trio is defined by

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 2\partial_x & \partial_x u^1 \\ u^1\partial_x & u^2\partial_x + \partial_x u^2 \end{pmatrix}, \tag{1}$$

$$R_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x^2. \tag{2}$$

The corresponding Monge metrics are

$$(\bar{g}_{1,ab}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (\bar{g}_{2,ab}) = \begin{pmatrix} 2u^2 & -u^1 \\ -u^1 & 2 \end{pmatrix}. \tag{3}$$

Classification: $n = 2$

The compatibility conditions $[P_1, R_2] = 0$ can be completely solved. The Monge metric of P_1:

$$\begin{aligned} \bar{g}_{11} &= c_0(u^2)^2 + c_3u^2 + c_4, \\ \bar{g}_{12} &= -c_0u^1u^2 - \frac{1}{2}c_3u^1 - \frac{1}{2}c_1u^2 + c_5, \\ \bar{g}_{22} &= c_0(u^1)^2 + c_1u^1 + c_2 \end{aligned}$$

The metric of P_1 is flat iff $c_0 = 0$, and linear for every value of the parameters: every two metrics in that space yield compatible nonlocal operators.

A speculation (might be wrong): compatibility can be regarded as a special position of the algebraic varieties involved.

Classification: $n = 4$

- We have a complete list of solutions to $[R_2, P_1] = 0$, with 288 cases (including the degenerate cases).
- Compatibility conditions are equivalent to a large system of algebraic equations.
- Fixing P_1 =const. we obtain 64 cases of solutions of $[R_2, P_2] = 0$, $[P_1, P_2] = 0$. Trios are parametrized by a finite number of parameters.
- The mutual geometric position of the trio of operators is much more difficult to understand.

Plücker's embedding for bi-Hamiltonian trios

(Lorenzoni, V. JPA 2024; Lorenzoni, Opanasenko, V. 2024)
There is a bijective correspondence between:

- bi-Hamiltonian trios of Hamiltonian operators as discussed:

$$A_1 = P_1, \qquad A_2 = Q_1 + R_2$$

- trios of two quadratic line complexes $\mathcal{P}_1$, $\mathcal{Q}_1$ and one linear line congruence $\mathcal{R}_2$; the varieties should be in a mutual position yet to be understood.

Third-order operators produce similar results (work in progress, Opanasenko, Lorenzoni, V.).

Perspectives

Hamiltonian operators	**Projective Geometry**
Third-order Hamiltonian operator	Quadratic Line Complex
Second-order Hamiltonian operator	System of n Linear Line Comp.
R_2-comp. first-order Hamiltonian op.	Quadratic Line Complex
R_3-comp. first-order Hamiltonian op.	???

Triples P_1, Q_1, $A_{2/3}$ and pairs P_1, $A_{2/3}$ of compatible operators are invariant under projective reciprocal transformations (provided we allow for nonlocal Ferapontov operators in the orbit).

The projective-geometric invariance of the corresponding hierarchies has *implications that are yet to be understood.*

Thank you!

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